Math X01: Introduction to Proofs

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Week 1 Plan

Note: This is taken from my vision of teaching an introduction to proofs course. This has yet to be implemented. The set up is the course meets three times per week, for **50** minutes each meeting.

1 Monday Part 1: Syllabus and Class Expectations

(10 min.) Briefly describe the class expectations (attendance, assessments, and schedule), the format of the course (Mondays and Wednesdays are lecture-based, Fridays are activity-based), and the course objectives:

Ω1) Understand what a mathematical proof is;

Ω2) Follow the logical steps of a mathematical proof;

Ω3) Read and write proofs of different techniques; and

Ω4) Read and write the symbols used by mathematicians for shorthand.

2 Monday Part 2: What is a Mathematical Proof?

(10 min.) **Passive Lecture**

- A proof is a rigorous and logical argument that demonstrates the truth of a statement.
- A proof are made up of these components:
  - Statements: a declarative sentence that definitively holds either a truth or false value. For example, I am six foot tall.
  - Axioms: a fundamental, agreed-upon truth that serves as the basis for reasoning. For example, a statement cannot be both true and false at the same time is an axiom.
  - Logical steps: sequence of arguments used to move from one statement to another.
  - Conclusion(s): the final result that logically resulted from the previous statements and logical steps.

(10 min.) **Active Lecture**

- A suggestive example from the past: Solve the following equation for x: $3x^2 - 27 = 0$.
  - Lay out the steps of solving this equation.
  - You may not realize it, but this is a type of mathematical proof. We have our axioms about arithmetic and numbers. Each line is a statement, and how we got each line is a logical step.
  - The conclusion is that $x = \pm 3$.
- Another example from the past: Solve the following equation for x: $x^2 - 7x + 12 = 0$.
  - Factoring or using quadratic formula.
  - By factoring, you get $(x - 3)(x - 4) = 0$. To reach the conclusion, we need to rely on a known theorem about the real numbers: whenever the product of two real numbers is zero, one of the factors must have been zero. This is the **Zero Product Principle of Real Numbers**. With that logical argument, we can reach our conclusion!
  - By quadratic formula, you are also using an established theorem about quadratic equations, a logical argument!
  - The conclusion is that $x = 3$ or $x = 4$.

(15 min.) **Passive Lecture**, point out the components of proof!

- Proof of the **Zero Product Principle of Real Numbers**:

  Suppose $a \cdot b = 0$ where a, b are real numbers.
  If $a = 0$, then we're done.
  Otherwise, then $a \neq 0$. Since a is a real number, it has a multiplicative inverse that is also a real number, let's call it c. Multiplicative inverse means that $a \cdot c = 1 = c \cdot a$. So, by multiplying c to both sides of our supposed equation, we have

$$
\begin{aligned}
c \cdot (a \cdot b) &= c \cdot 0 \\
(c \cdot a) \cdot b &= c \cdot 0 \\
1 \cdot b &= 0 \\
b &= 0
\end{aligned}
$$

  Thus, $b = 0$ and we're done. *Note: point out the logical arguments that let's us go from one line to the next: associativity, identity, multiplying by zero.*

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(5 min.) **Closing Remarks**

Recap the course objectives and how we've worked through some of them already just today. Next class, we will go through a crash course on some of the symbols mathematicians use for shorthand and we'll use throughout the course. Then, we'll talk about the first type of logical argumentation of the course, Propositional Logic.

3 Wednesday Part 1: Mathematical Symbols

(15 min.) **Passive Lecture**

- Set notation: $3 \in \{x : x \text{ is an odd number }\}$
- Famous sets: $\varnothing, \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.
- Proof shorthands: (to be given as a handout on Friday)

Shorthand	Meaning
$\forall$	for all
$\exists$	there exists
$\exists!$	there exists exactly one
$\nexists$	there does not exists
$\therefore$	therefore, thus, so
$\because$	because (or b/c)
s.t.	such that
WTS/NTS	want to show/need to show
WLOG	without loss of generality
$A \implies B$	statement A implies statement B
$A \impliedby B$	statement A is implied by statement B
$\iff$	if and only if (or iff)
$\implies \impliedby$	contradiction reached
$\square$ (or $\blacksquare$)	QED = quod erat demonstradum, end of proof

4 Wednesday Part 2: Propositional Logic

(10 min.) **Passive Lecture**, based on MRWP 1.1

- Propositional logic is logic that plays around with statements or propositions.
- Conventional starting example is: P = "The sky is blue." and Q = "It is raining." Note that P and Q are statements, they must either be true or false!
- When we string them like "if P, then Q", we get a conditional statement. This means that Q must be true whenever P is true, in other words $P \implies Q$.
  - The leading statement P is the hypothesis, while the receiving statement Q is the conclusion.
- Propositional logic happens when we examine the truth value of things stringing these statements P, Q, and so on with each other.
- *Teaser:* On Friday, we'll build truth tables using compounding statements. But, for today, let's explore some truth values of simple English statements.

(20 min.) **Active Lecture**

- (Adapted MWRP's Beginning Activity 2: Conditional Statements from Section 1.1) Consider the conditional statement: "If the fridge is empty, then Johan is hungry." What is the truth value of each statement below:
  - The fridge is empty AND Johan is hungry.
  - The fridge is empty and Johan is NOT hungry.
  - The fridge is NOT empty and Johan is hungry.
  - The fridge is NOT empty and Johan is NOT hungry.
- What are ways we can determine the truth value?
  - The first step is to make *conjectures* or guesses.
  - We can then use our **prior knowledge** to see if we can answer the truth-ness of our conjectures.
  - We can also use **examples**. If it just asks if something is possible, we can find one instance where it holds and conclude truth. If it asks if something will always hold true, we may try to find one instance where it doesn't hold true to conclude a contradiction.
  - Most importantly, **working with others** is key to brainstorming conjectures and arguments.
- Identify the truth value of each statement:
  - $(a^2 + b^2) = a^2 + b^2$ for all real numbers a and b.
  - There are integers x and y such that $2x + 5y = 41$.
  - If a is an integer, then a^2 is an even integer.
  - If a is an even integer, then a^2 is an even integer.

(5 min.) **Closing Remarks**

On Friday, we will explore Propositional logic some more using symbols next time. For now, you should know that we'll string these propositions using the following operations: $\wedge$ (and), $\vee$ (or), and $\neg$ (negate). For example, as a teaser, what is the truth value of the compounded statement:

(The fridge is empty **OR** Johan has **NOT** eaten take-out in a week) **AND** Johan is hungry.

5 Friday Part 1: Truth Tables

(25 min.) **Activity** (Activities are on the two pages)

Guide students through the first two columns of Table 1. Questions to ask at each step of completion:

- After Table 1: Without making the table, do you think $\neg P \implies \neg Q$ and $\neg(P \implies Q)$ have the same truth table?

Note: The contrapositive of $P \implies Q$ is $\neg Q \implies \neg P$. We will revisit this again later.

- After Table 2: What do you notice about this table?

Note: De Morgan's Law: $\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q)$ and $\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q)$. We will revisit this again later.

- After Table 3: I implore you to try $P \vee (Q \wedge R)$ and see if associativity is nice.

6 Friday Part 2: Good versus not-so-good Proofs

(12 min.) **Activity**

- Explain the premise of the activity. Emphasize that they don't need to know how to prove something about numbers yet.
- Walk around during activity for any questions people have.

(8 min.) **Debrief Activity** What made some of the proofs good while some are not-so-good?

(5 min.) **Weekly Debrief & Closing Remarks**
This week, we learned the components of a proof, saw some examples of them, and played around with Propositional Logic. You should at this point start understanding some of the symbols we use in mathematical proofs. You should also begin to develop an internal measure for what proofs are good and which ones need som more work.

Truth Tables

For the following conditional statements, fill out the table.

Table 1

P	Q	$\neg P$	$P \implies Q$	$\neg P \implies \neg Q$
T	T			
T	F			
F	T			
F	F			

Table 2

P	Q	$P \wedge Q$	$P \vee Q$	$(\neg P) \vee (\neg Q)$
T	T			
T	F			
F	T			
F	F			

Table 3

P	Q	R	$(P \vee Q) \wedge R$
T	T	T	
T	F	T	
F	T	T	
F	F	T	
T	T	F	
T	F	F	
F	T	F	
F	F	F	

Good versus not-so-good Proofs

Read the following theorem:

Theorem: The product of two odd integers is an odd integer.

Conceptual Review Question: What is the hypothesis and conclusion here?

Now, read the four different proofs provided to you. Your group's task is to grade (out of a maximum of four points) these manufactured proofs by how understandable and sound their logic is. Imagine that you are the instructors having to grade students' work on an exam question that asks to prove the theorem above. The goals of the task is to get a feel for what makes some proofs good and what makes them not-so-good.

Proof 1: Suppose a and b are odd integers, so they look like $a = 2n + 1$ and $b = 2m + 1$. Multiplying, we get $ab = 4nm + 2m + 2n + 1$. Since it ends with $+1$, it must be odd.

Proof 2: 3 times 5 is 15 which is still odd, so it must hold.

Proof 3: Suppose a and b are odd integers. Let $a = 2n + 1$ and $b = 2m + 1$, for some integers n, m. Multiplying them, we get $ab = 4nm + 2m + 2n + 1$. By factoring, we get $ab = 2(2mn + m + n) + 1$. Since $2mn + m + 1$ is a sum of integers, it must be an integer. Thus, the product ab is odd because it is of the form $2k + 1$ where k is an integer.

Proof 4: By contrapositive, suppose a is odd and b even. Let $a = 2n + 1$ and $b = 2m$, for some integers n, m. Multiplying them, we get $ab = 4mn + 2m$. By factoring, we get $ab = 2(2mn + m)$. Since $2mn + m$ is an integer, we have $ab = 2k$ for some integer k. Thus, ab is an even number. So, the contrapositive gives us the theorem.